Theorem:-

Intersection of two subrings of a ring R is a subring of that ring.

$\Rightarrow$ Let S_1 and S_2 be two subring of a ring R.

Since $0 \in S_1$ and $0 \in S_2$

$\Rightarrow 0 \in S_1 \cap S_2$

so, $S_1 \cap S_2 \neq \phi$.

let $a, b \in S_1 \cap S_2$ be arbitrary.

$\Rightarrow a, b \in S_1$ and $a, b \in S_2$

$\Rightarrow a-b \in S_1$ and $a-b \in S_2$ [since S_1, S_2 are subring]
$a \cdot b \in S_1$ $a \cdot b \in S_2$

$\Rightarrow a-b \in S_1 \cap S_2$
$a \cdot b \in S_1 \cap S_2$

Since a, b are arbitrary.
So, $S_1 \cap S_2$ is a subring of R.

Note:→
The union of two subrings may not be a subring of R.
For example, $(2\mathbb{Z}, +, \cdot)$ and $(3\mathbb{Z}, +, \cdot)$ are subring of the ring $(\mathbb{Z}, +, \cdot)$ but $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subring of $\mathbb{Z}$, infact $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subgroup of $\mathbb{Z}$.

Ideal of a Ring:–

A subring S of a ring R is said to be
(i) a left ideal of R if $a \in S, r \in R \Rightarrow r.a \in S$.
(ii) a right ideal of R if $a \in S, r \in R \Rightarrow a \cdot r \in S$.
(iii) a both sided ideal (i.e. an ideal) of R if $a \in S, r \in R \Rightarrow a \cdot r \in S$ and $r.a \in S$.

Result:
Let R be a ring and S be a non-empty subset of R. Then S is an ideal of R iff
(i) $a, b \in S \Rightarrow a-b \in S$
(ii) $a \in S, r \in R \Rightarrow a \cdot r \in S$ and $r.a \in S$.

Proof: Let S be an ideal of R. Then S is a subring of R such that $a \in S, r \in R \Rightarrow r.a \in S$ and $a \cdot r \in S$.
Since S is a ring, $a \in S, b \in S \Rightarrow a-b \in S$
Therefore both the condition (i) and (ii) holds.

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Conversely, let S be a non-empty subset of R such that (i) and (ii) both hold.

Since (i) holds, $(S, +)$ is a subgroup of the group $(R, +)$

Since (ii) holds, $a \in S, b \in S \Rightarrow a \in S, b \in R \Rightarrow a \cdot b \in S$

Therefore S is a non-empty subset of R such that $(S, +)$ is a sub-group of the group $(R, +)$ and $a \in S, b \in S \Rightarrow a \cdot b \in S$.

Hence S is a subring of R.

Since S is a subring of R and (ii) holds so S is an ideal of R.

Examples :-

(i) $(\mathbb{Z}, +, \cdot)$ is a subring of $(\mathbb{Q}, +, \cdot)$ but not an ideal of $(\mathbb{Q}, +, \cdot)$ because let,

$R = (\mathbb{Q}, +, \cdot)$, $S = (\mathbb{Z}, +, \cdot)$

Then $\frac{2}{3} \in R$ and $2 \in S$

So $2 \cdot \frac{2}{3} = \frac{4}{3} \notin S$ hence S is not an ideal of R.

(ii) Every subring of $\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

(iii) Every subring of $\mathbb{Z}_n$ is an ideal.

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